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Symmetry and Localization in Noetherian Prime PI Rings

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For a prime ideal P in a prime Noetherian PI ring, a finite criterion is given for P to be left or right localizable. One consequence of this is that P is right localizable if and only if it is left localizable. A corollary is that if S is a simple module over such a ring and for all simple modules T, $\mathrm{Ext}(S, T) \neq 0$ implies $S \cong T$, then for all simple modules T, $\mathrm{Ext}(T, S) \neq 0$ implies $S \cong T$. This is a special case of a general symmetry result concerning the graph of links of a prime Noetherian PI ring. A consequence of this symmetry is that if S is any subset of R satisfying the right (or left) Ore condition, then the corresponding localization of R is actually a two-sided localization with respect to a two-sided Ore set. Another consequence is that an ideal in such a ring has the left AR (Artin–Rees) property if and only if it has the right AR property.

INTRODUCTION

For any prime ideal P in a Noetherian ring R, we let $\mathscr{C}(P)$ be the set of elements regular modulo P. We say that P is *right* (or left) *localizable* if this set satisfies the right (or left) Ore condition, and that it is *localizable* if it

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satisfies both the right and left Ore conditions. The question of finding a criterion for localizability of a prime ideal was posed by Goldie in [6].

We recall that if R is a prime PI ring, then it is contained in a larger subring of its quotient ring, the *trace ring* of R, written $T(R)$. If R is Noetherian, then $T(R)$ is a finite central extension of R [1, 2.4], and is integral over its center.

DEFINITION. If P and P_1 are primes of R, then P and P_1 are *tr-linked* if there are primes Q and Q_1 of $T(R)$, such that $Q \cap R = P$ and $Q_1 \cap R = P_1$, and such that $Q \cap Z = Q_1 \cap Z$, where Z is the center of $T(R)$.

Our criterion for the localizability of P is then the following:

THEOREM A. *If P is a prime ideal in a Noetherian prime PI ring R, then the following three conditions are equivalent:*

(i) *P is left localizable.*

(ii) *P is right localizable.*

(iii) *If Q is a prime which is tr-linked to P, then $Q = P$.*

To see that this is a finite criterion for localizability, let $P_1, ..., P_r$ be the primes in $T(R)$ such that $P_i \cap R = P$, and let $p_i = P_i \cap Z$. Then to check the above criterion one must verify that if Q is a prime in $T(R)$ with $Q \cap Z = p_i$ for some i, $1 \leqslant i \leqslant r$, then $Q = P_j$ for some j, $1 \leqslant j \leqslant r$. By [2, Proposition 5], there are at most nr prime ideals in $T(R)$ contracting to one of the ideals p_i (where n is the PI-degree of R). Thus there are only nr ideals to check in verifying this condition.

The equivalence of (i) and (ii) in Theorem A generalizes the result of [3], where the equivalence is proved under the additional asumption that R is a finite module over its center. Since Theorem A says that $\mathscr{C}(P)$ is a right Ore set if and only if it is a two-sided Ore set, it is a special case of the following result, which says that for prime Noetherian PI rings, right and left localizations are invariably the same.

THEOREM B. *If R is a Noetherian prime PI ring, and S is a right (or left) Ore set in R, then there is a two-sided Ore set C such that $C \supseteq S$ and the ring of right (or left) fractions of R obtained from S is the same as the (two-sided) ring of fractions obtained from C.*

This, of course, suggests that an even stronger result might hold—that right (or left) Ore sets might necessarily be two-sided Ore sets. This is proved under special circumstances in [3], but is unknown in general.

While both of the results above have quite classical statements, the proofs require that we consider more recent ideas relating the Ore condition to the representation theory of the ring. To understand the representation theoretic interpretation of Theorem A, we need to consider another notion of link between primes. If P and Q are prime ideals in a Noetherian PI ring, then we say there is a *link* $P \rightsquigarrow Q$ if r-ann$(P \cap Q/PQ) = Q$ and l-ann$(P \cap Q/PQ) = P$. (For Noetherian rings in general, a more careful definition is required—this description, from [13], is valid for fully bounded Noetherian rings.) If P and Q are maximal ideals, with corresponding simple right modules S and T, then $P \rightsquigarrow Q$ if and only if $\operatorname{Ext}(S, T) \neq 0$. (If S' and T' are the corresponding simple left modules, then $P \rightsquigarrow Q$ if and only if $\operatorname{Ext}(T', S') \neq 0$.) The prime ideals together with these links form the *graph of links* of the ring. The connection between the graph of links and the Ore condition is provided by the following fundamental fact: *If $P \rightsquigarrow Q$ and if S is a multiplicatively closed set satisfying the right Ore condition, and $S \subseteq \mathscr{C}(Q)$, then $S \subseteq \mathscr{C}(P)$* [9, 5.4.4]. (Symmetrically, if S is a multiplicatively closed set satisfying the left Ore condition, and $S \subseteq \mathscr{C}(P)$, then $S \subseteq \mathscr{C}(Q)$.) We say that a subset X of $\operatorname{Spec}(R)$ is *right link closed* if $P \in X$ whenever $Q \in X$ and $P \rightsquigarrow Q$. We apologize for the apparent backwardness of this terminology, but it has the virtue that if S is a right Ore set, then the set of primes P such that $\mathscr{C}(P) \supseteq S$ is a right link closed set. Left link closed sets are defined dually, with the corresponding conclusion. A set which is both right and left link closed is simply called *link* closed. For any prime ideal P, the *clique* of P, written $\operatorname{Cl}(P)$, is the smallest left and right link closed subset of $\operatorname{Spec}(R)$ containing P. To fix terminology, we will call a set X of primes *tr-closed* if every prime which is *tr*-linked to a prime in X is also in X.

If R is a Noetherian PI ring (or a fully bounded Noetherian ring) and P is a prime ideal, then P is right localizable if and only if the set $\{P\}$ is right link closed [9, 7.1.5]. Thus, combining Theorem A with the interpretation given above for maximal ideals, we obtain the next result.

THEOREM C. *If R is a Noetherian prime PI ring and S a simple right module, then the following conditions are equivalent*:

(i) *For all simple modules T,* $\operatorname{Ext}(S, T) \neq 0$ *implies $S \cong T$.*

(ii) *For all simple modules T,* $\operatorname{Ext}(T, S) \neq 0$ *implies $S \cong T$.*

(iii) *If P is the annihilator of S, then P is a localizable maximal ideal.*

This is really a special case of the following more general theorem.

THEOREM D. *If R is a Noetherian prime PI ring, and X a subset of* $\operatorname{Spec}(R)$, *then the following properties of X are equivalent*:

(i) *X is left link closed,*

(ii) *X is right link closed,*

(iii) *X is link closed,*

(iv) *X is tr-closed.*

COROLLARY TO THEOREM D. *If R is a Noetherian prime PI ring and X is a set of simple modules over R, then the following properties of X are equivalent*:

(i) *If* $S \in X$ *and* $\operatorname{Ext}(S, T) \neq 0$ *then T is isomorphic to an element of X.*

(ii) *If* $S \in X$ *and* $\operatorname{Ext}(T, S) \neq 0$ *then T is isomorphic to an element of X.*

(iii) *The set of annihilators of the elements of X is a tr-closed family of maximal ideals.*

We remark that this applies in particular to the representations of a finite-dimensional Lie algebra over a field of characteristic p, $p > 0$, since the enveloping algebra of such a Lie algebra is a prime Noetherian PI ring. It also applies to the restricted enveloping algebra of a restricted Lie algebra, since this is obtained from the ordinary enveloping algebra by factoring out a polycentral ideal, and this process does not alter links between distinct prime ideals (compare [16, Lemma 5].) (This fact about the restricted enveloping algebra can also be obtained from the known fact that it is a Frobenius algebra.)

The behavior described above, however, is quite different from what obtains in the nonprime case or for other classes of Noetherian rings. For example, the link graph of the upper triangular matrix ring

$$\begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$$

(for a field k) consists of two points with a single arrow between them. Clearly none of the above results holds for this ring. More interestingly, if U is the enveloping algebra of the two-dimensional non-Abelian solvable Lie algebra (over the complex numbers), then the graph of links of U has many infinite components (consisting of maximal ideals) containing no nontrivial cycles. (If the algebra is generated by elements y and x with $[y, x] = x$, then the maximal ideals are of the form $M_\alpha = (x, y + \alpha)$, and there is a link $M_\alpha \rightsquigarrow M_\beta$ if and only if $\beta = \alpha$ or $\beta = \alpha + 1$.) Clearly, such a ring has many left link closed sets which are not right link closed.

The symmetry results above suggest other possible results which, however, examples easily show to be false. We refer to [5], where there is a discussion of the graph of links of "tiled orders," which already enables one to construct a variety of examples of finite graphs which appear as link graphs. The best examples of infinite connected components in the graphs of links of PI rings appear in [14].

In the first section we explore the connection between links in a prime Noetherian PI ring and its trace ring, obtaining results on localizations of the trace ring and Theorem A above. In the second section we consider arbitrary left or right Ore sets, obtaining Theorems D and B.

1. Localization at Prime and Semiprime Ideals

If X is a subset of Spec(R), then $\mathscr{C}(X)$ is defined to be $\bigcap\{\mathscr{C}(P)\colon P\in X\}$. The present point of view in localization theory is that to find the correct notion of "localization at a prime ideal," one should take a prime ideal P, and then look at the clique of P, Cl(P), because if S is a (two-sided) Ore set contained in $C(P)$, then $S\subseteq\mathscr{C}(\mathrm{CL}(P))$. If R is a Noetherian PI ring and Cl(P) is finite, then Cl(P) is a localizable set of primes [13, Theorem 5]. (In more traditional language, the intersection of this finite set of primes is a localizable semiprime ideal.) If R is a finite module over its center, then Müller shows [12, Theorem 7] that a clique consists precisely of those primes whose intersection with the center is a given prime ideal of the center, and it follows that all cliques are finite and this theory applies. This also holds if $R=T(R)$ as we will see below. (It seems to be unknown whether this holds for every prime ring integral over its center. That is, it is clear that for such a ring the cliques are finite, but it is unknown whether they are in one-to-one correspondence with the primes of the center.)

Lemma 1. *Let R be a prime Noetherian PI ring and let S be a right Ore set in R. Then*

$$T(R_S)=T(R)_S=T(R)[1/\det(s)\colon s\in S].$$

Proof. This is a combination of Lemmas 1 and 2 in [3].

In the following, we will use the notion of Krull dimension due to Gabriel and Rentschler (for an exposition cf. [7].) If R and S are Noetherian PI rings and B an $R-S$-bimodule which is finitely generated on each side, then it follows from [8] that K. dim.$(R/\text{l-ann}(B))=$ K. dim.$(S/\text{r-ann}(B))$. Hence if P and Q are in the same clique, then K. dim. $R/P=$ K. dim. R/Q. We will call K. dim. R/P the *codimension* of P. We recall that there is an ordinal-valued notion of classical Krull dimension as well (cf. [11] or [7, pp. 48–49]), which gives another notion of codimension for a prime ideal, which we will call *classical codimension.* Thus our notation will say that cl. codim.$(P)=$ cl. K. dim.(R/P). We use the fact that for Noetherian PI rings (and, more generally, fully bounded Noetherian rings), the notions of classical codimension and codimension coincide [11; 7, p. 58]. The two notions will be necessary, since we will use the classical codimension for a non-Noetherian commutative ring.

LEMMA 2. *If P is a prime ideal of R (a prime Noetherian PI ring), if Q is a prime ideal of $T(R)$ contracting to P, and $p = Q \cap Z$, where Z is the center of $T(R)$, then*

$$\text{K. dim. } R/P = \text{K. dim. } T(R)/Q = \text{cl. K. dim. } Z/p.$$

Proof. We may regard $T(R)/Q$ as an $R/P - T(R)/Q$-bimodule. Since $T(R)$ is a finite centralizing extension of R, this bimodule is Noetherian on each side, and it follows [8] that K. dim. $R/P =$ K. dim. $T(R)/Q$. We next note that since $T(R)$ is integral (though not necessarily finite) over Z, it is routine that primes in these two rings satisfy LO, GU, and INC (as in [10, pp. 27–31]). Thus cl. codim. $Q =$ cl. codim. p, by an easy induction, and the result follows from the fact that K. dim. $T(R)/Q =$ cl. K. dim. $T(R)/Q$.

PROPOSITION 3. *If R is a prime Noetherian PI ring and $R = T(R)$, then every clique of primes in R is finite. Further, for a given prime P of R, the following three sets are the same:* (i) Cl(P), (ii) *the prime ideals Q such that $P \cap Z = Q \cap Z$, where Z is the center of R, and* (iii) *the smallest right link closed subset of* Spec(R) *containing P.*

Remark. We note that this shows that R has a complete localization theory, in the sense that for every prime ideal P, there is an Ore set such that P corresponds to a maximal ideal in the resulting localization, and the localizations are all semilocal rings. It also follows, just as with Müller's result [12] concerning Noetherian rings finitely generated as modules over their centers, that all of the localizations are central. (Even stronger—in $T(R)$ all one-sided localizations are central.)

Proof. Let P be a prime ideal of R and let $p = P \cap Z$. Since p is a prime ideal, we can localize at the prime p. The primes in R corresponding to maximal ideals in the localized ring R_p are precisely the primes P' in R such that $P' \cap Z = p$. (Here we use INC to show that these primes are incomparable.) Since these primes are all minimal over pR, there are only finitely many of them. Let us call this set of primes X. Lemma 2 implies that if $P' \in X$, then K. dim. $R/P' =$ K. dim. R/P. The fundamental property of cliques shows that if Q is in Cl(P), then R/Q is torsion-free with respect to the Ore set $Z - p$, and hence $Q \subseteq P'$ for some $P' \in X$. Since $P \rightsquigarrow Q$ implies that K. dim. $R/P =$ K. dim. R/Q, it follows that $Q = P'$, so Cl$(P) \subseteq X$. Hence Cl(P) is finite. Let X' be the subset of CL(P) which is the smallest right link-closed subset of Spec(R) containing P. (That is, $Q \in X'$ if there are primes $Q_1, ..., Q_n$ in R with $Q = Q_1$, $P = Q_n$, and $Q_i \rightsquigarrow Q_{i+1}$ if $1 \leqslant i < n$.) Since X' is finite, it is right localizable [9, 7.1.5]. If R' is the localization of R at X', then the maximal ideals of R' are in one-to-one correspondence with the primes in X'. However, Lemma 1 shows that this localization is central, so that if $Q \cap Z = P \cap Z$, then $QR' \neq R'$. Hence $Q \in X'$, and we conclude that $X' = \mathrm{CL}(P) = X$, which is the statement of the proposition.

LEMMA 4. *Let R be a prime Noetherian PI ring and S a right Ore set in R. If P is a prime ideal of R such that $S \subseteq C(P)$, and P is tr-linked to P_1, then $S \subseteq C(P_1)$.*

Proof. We use frequently the fact that if S is a right Ore set in R and Q a prime ideal, then R/Q is either torsion-free or torsion with respect to S. The statement that $S \subseteq \mathscr{C}(P)$ means precisely that R/P is torsion-free with respect to S. Now if Q is a prime ideal of $T(R)$ such that $Q \cap R = P$, then $T(R)/Q$ must be torsion-free with respect to the Ore set S, so $S \subseteq \mathscr{C}(Q)$. (Here we use the fact that S is an Ore set in $T(R)$, which follows from the fact that $T(R)$ is centrally generated over R.) Let $p = Q \cap Z$, where Z is the center of $T(R)$. If Q_1 is another prime ideal of $T(R)$ such that $Q_1 \cap Z = p$, then according to Lemma 3, Q_1 is in the smallest right link closed subset of Spec($T(R)$) containing Q, so $S \subseteq \mathscr{C}(Q_1)$. Finally, if $P_1 = Q_1 \cap R$, then since $S \subseteq R$, we have $S \subseteq \mathscr{C}(P_1)$.

Our main results depend on the following proposition which compares the graph of links of a ring R and a finite centralizing extension of R (i.e., a langer ring generated as an R-module by a finite set of elements which commute with the elements of R.)

PROPOSITION 5. *Let R be a Noetherian PI ring and S a finite centralizing extension of R. Let P and Q be distinct prime ideals of R with $P \rightsquigarrow Q$. Then there exist prime ideals $L_1, ..., L_t$ in S such that $L_1 \cap R = P$, $L_t \cap R = Q$, and $L_i \rightsquigarrow L_{i+1}$ for $i = 1, ..., t-1$. Similarly, if P and Q are prime ideals of S with $P \rightsquigarrow Q$, then either $P \cap R = Q \cap R$ or $P \cap R \rightsquigarrow Q \cap R$.*

Remark. For those interested in such questions, we remark that the assumption that R is a PI ring can be replaced (using almost the same argument) with the assumption that R satisfies the second layer condition (as defined in [9]). We are indebted to B. J. Müller for pointing out that a result very similar to Proposition 5 is contained in the unpublished dissertation of his student J. C. Royle [17].

Proof. We first suppose that P and Q are prime ideals of R with $P \rightsquigarrow Q$. Since $P \rightsquigarrow Q$, we know that r-ann$(P \cap Q/PQ) = Q$ and l-ann$(P \cap Q/PQ)$

$=P$. If A/PQ is the left torsion submodule of $P \cap Q/PQ$, then using the previously cited result of [8] concerning the left and right Krull dimensions of bimodules, it is easy to verify that $A \neq P \cap Q$, and that for every ideal I such that $P \cap Q \supseteq I \supseteq A$ with $I \neq A$ we have l-ann$(I/A) = P$ and r-ann$(I/A) = Q$. (This, by the way, is the definition of link used in the more general theory; cf. [9].)

Case (i). We first assume that R is Artinian. We have a sequence of $R \otimes R^{\text{op}}$-modules

$$S \supseteq R \supseteq P \cap Q \supseteq A \supseteq 0,$$

where $P \cap Q \neq A$. We let

$$S = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_n = 0$$

be a composition series of S as a left $S \otimes S^{\text{op}}$-module, so that if $M_i =$ l-ann(V_i/V_{i+1}) and $N_i =$ r-ann(V_i/V_{i+1}), then M_i and N_i are maximal ideals in S for $i = 0, ..., n-1$. We now apply the Schreier refinement theorem (or Jordan–Hölder theorem) to these two sequences of $R \otimes R^{\text{op}}$-submodules of S, thus obtaining an ideal I of R with $I \supseteq A$, such that I/A is a simple $R \otimes R^{\text{op}}$-module, and left $R \otimes R^{\text{op}}$-submodules W_1 and W_2 of S with

$$V_j \supseteq W_1 \supseteq W_2 \supseteq V_{j+1}$$

for some index j $(0 \leqslant j < n)$, such that $I/A \cong W_1/W_2$ (where the isomorphism is an $R \otimes R^{\text{op}}$-isomorphism). Consequently, l-ann(W_1/W_2) $= P$ and r-ann$(W_1/W_2) = Q$. Sinc W_1/W_2 is a subfactor of V_j/V_{j+1}, it follows that $P \supseteq M_j \cap R$ and $Q \supseteq N_j \cap R$. Since M_j and N_j are maximal ideals in S, it follows from GU that $M_j \cap R$ and $N_j \cap R$ are maximal ideals of R, so $P = M_j \cap R$ and $Q = N_j \cap R$.

To complete the argument in this case (when R and S are Artinian) we must show that there are maximal ideals $L_1, ..., L_t$ of S such that $L_1 = M_j$, $L_t = N_j$, and $L_i \rightsquigarrow L_{i+1}$ for $1 \leqslant i < t$. This is a standard fact, but we include a proof. Let J be a right ideal of S, maximal with respect to the property that $J \cap V_j \subseteq V_{j+1}$. Thus, S/J will be an essential extension of the semisimple right module V_j/V_{j+1}. If

$$S/J = S_1 \supseteq S_2 \supseteq \cdots \supseteq S_m = 0$$

is a composition series for the right S-module S/J, and r-ann$(S_i/S_{i+1}) = U_i$, $1 \leqslant i < m$, then

$$(S/J)\, U_1 U_2 \cdots U_{m-1} = 0,$$

whence it follows that $J \supseteq U_1 U_2 \cdots U_{m-1}$. This implies that $SU_1 U_2 \cdots$

$U_{m-1} \cap V_j \subseteq V_{j+1}$, so V_j/V_{j+1} is an ideal subfactor of $S/U_1 U_2 \cdots U_{m-1}$. We infer that

$$U_1 U_2 \cdots U_{m-1}(V_j/V_{j+1}) = 0.$$

Hence $M_j \supseteq U_1 U_2 \cdots U_{m-1}$, and thus $M_j = U_i$ for some index i, $1 \leqslant i < m$. The statement now follows easily from the "Ext" description of links.

Case (ii). We now consider the general case. Let B be a right R-submodule of S maximal with respect to the property that $B \cap (P \cap Q) = A$. Hence (as in the previous argument) S/B is an essential extension of $P \cap Q/A$ (as right R-modules). By [8], there are prime ideals $Q_1, ..., Q_s$ in $\mathrm{Cl}(P)$ so that

$$(S/B)\, Q_1 \cdots Q_s = 0.$$

Let $I = Q_1 \cdots Q_s$. Then $B \supseteq SI$, so we may work with $S' = S/SI$, $R' = R/SI \cap R$. (Note that SI is an ideal of S since S is a centralizing extension of R.) Since, as in the previous argument, $P \cap Q/A$ is an $R-R$-bimodule subfactor of S/SI, it follows that $P \cap Q \supseteq I$, so we may replace P and Q by $P' = P/I$ and $Q' = Q/I$. Now the minimal primes of R' are all of the form Q_i/I for some $Q_i \in \mathrm{Cl}(P)$. Hence the minimal primes of R' all have the same codimension as P', from which we conclude that $\mathrm{Cl}(P')$ consists entirely of minimal primes of R', and thus is finite. (Note that the fact that two primes are linked in R does not *a priori* imply that their images in R' are linked, but primes linked in R' are necessarily linked in R.) It follows (from [13] or [9, 8.35]) that if $\mathscr{C} = \mathscr{C}(\mathrm{Cl}(P'))$, then $\mathscr{C}$ is a (left and right) Ore set, and $R'\mathscr{C}^{-1}$ is Artinian. We note that $S'\mathscr{C}^{-1}$ is a finite centralizing extension of $R'\mathscr{C}^{-1}$. Since $P' \rightsquigarrow Q'$, it follows easily that $P'\mathscr{C}^{-1} \rightsquigarrow Q'\mathscr{C}^{-1}$, and we can therefore apply the result from Case (i) to obtain a chain of maximal ideals L_i' of $S'\mathscr{C}^{-1}$ $(1 \leqslant i \leqslant t)$ such that $L_i' \rightsquigarrow L_{i+1}'$ $(1 \leqslant i < t)$ while $L_1' \cap R'\mathscr{C}^{-1} = P'\mathscr{C}^{-1}$ and $L_t' \cap R'\mathscr{C}^{-1} = Q'\mathscr{C}^{-1}$. If $v\colon S \to S'\mathscr{C}^{-1}$ is the natural map, then we obtain the desired primes L_i of S by setting $L_i = v^{-1}(L_i')$.

For the converse, suppose that P and Q are primes in S with $P \rightsquigarrow Q$ and let $P' = P \cap R$, $Q' = Q \cap R$. As before there is an ideal A such that $P \cap Q \supseteq A \supseteq PQ$ with $P \cap Q \neq A$, such that $P \cap Q/A$ is torsion-free as a right S/Q-module and as a left S/P-module. Without loss of generality we may assume that $A = 0$ (replacing S by S/A and R by $R/R \cap A$.) Now $P \cap Q/A$ is also an $R/P' - R/Q'$-bimodule, and as such is finitely generated and torsion-free on each side. It follows from [8] that $\mathrm{K.dim.}(R/P') = \mathrm{K.dim.}(R/Q')$. Hence either $P' = Q'$ or P' and Q' are incomparable. Since $P'Q' = 0$ it follows that P' and Q' are precisely the minimal primes of R. Since for any other prime ideal T of R we have $\mathrm{K.dim.}(R/T) <$

K. dim. $R(R/P)$, it follows that the set $\{P', Q'\}$ is a link closed set. If it were not true that $P' \rightsquigarrow Q'$, then $\{Q'\}$ would be right link closed, and Q' would be right localizable. If $\mathscr{C} = \mathscr{C}(Q')$, then $\mathscr{C}$ would be a right Ore set in S (since S is a centralizing extension of R), and since $\mathscr{C} \subseteq \mathscr{C}(Q)$ and $P \rightsquigarrow Q$, we would have $\mathscr{C} \subseteq \mathscr{C}(P)$. Since $\mathscr{C} \subseteq R$, we would have $\mathscr{C} \subseteq \mathscr{C}(P')$, which would contradict the right localizability of Q'. Hence $P' \rightsquigarrow Q'$ as required.

COROLLARY TO PROPOSITION 5. *If R is a prime Noetherian PI ring and P and Q are primes in R with $P \rightsquigarrow Q$ then P and Q are tr-linked.*

Proof. Proposition 5 shows that there are primes P_1 and Q_1 of $T(R)$ which restrict to P and Q and which are in the same clique in $\operatorname{Spec}(T(R))$. According to Proposition 3, we must have $P_1 \cap Z = Q_1 \cap Z$, which proves the statement.

Proof of Theorem A. Let P be a right localizable prime of R. If Q is a prime ideal of R, then $\mathscr{C}_R(Q) \supseteq \mathscr{C}_R(P)$ if and only if $P \supseteq Q$. Hence, according to Lemma 4, if Q is tr-linked to P, then $P \supseteq Q$. Since Lemma 2 implies that tr-linked primes have the same codimension, it follows that if Q is tr-linked to P, then $Q = P$. This proves that (i) implies (iii). The Corollary to Proposition 5 shows that linked primes are tr-linked, so we conclude that $\{P\}$ is a left link closed set as well, so P is left localizable, proving (ii). The reverse implications are symmetric.

Applying the same argument to finite link closed sets, one easily obtains the following generalization.

PROPOSITION 6. *A semiprime ideal in a prime Noetherian PI ring is right localizable if and only if it is left localizable, and this occurs if and only if the set of primes minimal over it is tr-closed.*

2. SYMMETRY AND TWO-SIDED LOCALIZATIONS

Our basic symmetry result is Theorem D, the proof of which is now easy. The statements on the left and right AR property and the fact that all localizations are two-sided will follow naturally.

Proof of Theorem D. It follows from the Corollary to Proposition 5 that conditions (i), (ii), and (iii) of Theorem D all imply condition (iv). We must therefore prove that a tr-closed subset of $\operatorname{Spec}(R)$ is link closed. We show that if P and P' are primes in the Noetherian prime PI ring R which are tr-linked, then there are prime ideals $P_0, ..., P_m$ of R with $P_0 = P$ and $P_m = P'$, such that $P_i \rightsquigarrow P_{i+1}$ for all i, $i = 0, ..., m-1$. (A symmetric argument shows that there are primes $T_0, ..., T_k$ of R with $T_0 = P'$ and $T_k = P$, such that $T_i \rightsquigarrow T_{i+1}$ for all i, $i = 0, ..., k-1$.)

Our condition implies that there are primes Q and Q' of $T(R)$ such that $Q \cap R = P$ and $Q' \cap R = P'$, such that $Q \cap Z = Q' \cap Z$, where Z is the center of $T(R)$. It follows from Proposition 3 that there are prime ideals $Q_0, ..., Q_m$ of $T(R)$ such that $Q_0 = Q$ and $Q_m = Q'$, such that $Q_i \rightsquigarrow Q_{i+1}$ for all i, $i = 0, ..., m-1$. If we let $P_i = Q_i \cap R$, then the second part of Proposition 5 implies that $P_i \rightsquigarrow P_{i+1}$ for all i, $i = 0, ..., m-1$, which proves the result.

COROLLARY 7. *If P and P' are prime ideals in a Noetherian prime PI ring and $P \rightsquigarrow P'$, then there are primes $P_1, ..., P_r$ such that $P' = P_1$, $P = P_r$, $P_i \rightsquigarrow P_{i+1}$ for all i, $i = 1, ..., r-1$, and $r \leqslant$ p.i. degree(R).*

Proof. Using the argument in the proof of Theorem D, we see that it is enough to establish the statement under the assumption that $T = T(R)$. In that case, such a sequence exists (by the proof of Theorem D), and we need only establish the final inequality. However, this follows from [2, Theorem 8], which asserts (in particular) that in $T(R)$, each clique has at most p.i. degree(R) elements.

COROLLARY 8. *If P and P' are prime ideals in a Noetherian prime PI ring of p.i. degree 2, and $P \rightsquigarrow P'$, then $P' \rightsquigarrow P$.*

If R is a ring and I an ideal, we say that I has the *right AR property* if for every finitely generated module M containing a essential submodule N with $NI = 0$, there is a positive integer n such that $MI^n = 0$. (A statement more similar to the usual commutative notion is the following: for every right ideal J there is a positive integer n such that $JI \supseteq J \cap I^n$.) For Noetherian PI rings, and more generally, for fully bounded Noetherian rings, it is an immediate consequence of the extension theory in [8] that an ideal I has the right AR property if and only if the set X defined by

$$X = \{P \in \operatorname{Spec}(R) \colon P \supseteq I\}$$

is right link closed. As a consequence of Theorem D, we obtain the following immediate corollary.

COROLLARY 9. *If R is a prime Noetherian PI ring and I an ideal in R, then I has the right AR property if and only if I has the left AR property.*

Though this proof is not valid for prime Noetherian rings which are not PI, there do not seem to be examples of ideals in prime Noetherian rings which have the AR property on one side only. For nonprime rings, such examples are common (for example, of the two primes in the ring of 2×2 upper triangular matrices over a field, one is right AR but not left AR, and the other is left AR but not right AR.)

We now turn to the main result of this part of the paper, which is that all rings of fractions of prime Noetherian PI rings are two-sided. A key point in proving Theorem B is to notice that any right (or left) Ore set in a Noetherian PI ring (or, more generally, a fully bounded ring) is essentially given by a set of prime ideals, in a way which we will now make clear.

If X is a subset of Spec(R), then (as in Section 1) $\mathscr{C}(X)$ is defined to be $\bigcap\{\mathscr{C}(P): P\in X\}$. We say that X is *right localizable* (or *left localizable*, or *localizable*) if (i) *$\mathscr{C}(X)$ satisfies the right Ore condition* (or left Ore condition, or both Ore conditions), and (ii) if $\mathscr{C}=\mathscr{C}(X)$, *then the primes in R which are the restrictions to R of the primitive ideals of $R\mathscr{C}^{-1}$ are precisely the primes in X.* (This second condition avoids certain absurdities—for example, if X is the set of all but one maximal ideal in $k[x, y]$, then $\mathscr{C}(X)$ is a right Ore set, because it is exactly the set of units of $k[x, y]$, but we would not want to say that X is localizable.) If $\mathscr{C}(X)$ satisfies condition (i), then condition (ii) is easily seen to be equivalent to the following "intersection condition" [15, 9]: (ii)′ *if I is a right ideal which contains an element of $\mathscr{C}(P)$ for every $P\in X$, then I contains an element of $\mathscr{C}(X)$.* (We should emphasize that (ii) and (ii)′ are redundant when the set X is finite.)

LEMMA 10. *Let R be an FBN ring and X a set of incomparable prime ideals in R. Then X is* (*right*) *localizable if and only if*

(i) *X is* (*right*) *link closed, and*

(ii) *for every ideal I, if $I\cap\mathscr{C}(P)\neq\varnothing$ for all $P\in X$, then $I\cap\mathscr{C}(X)\neq\varnothing$.*

Remark. We are indebted to Bruno Müller for essentially this remark, which improves on the standard localization result [4; 9, 7.1.5] in a key respect—that condition (ii) is symmetric.

Proof. We refer to the proof of a standard result of Jategaonkar along these lines ([4] or [9, 7.1.4(a)]). Since X is an incomparable and (right) link closed set of primes, the only one of the standard conditions which is lacking is the "intersection condition." An inspection of the proof shows that what is really required is that for every right ideal I, if R/I is not $\mathscr{C}(X)$-torsion, then for some $P\in X$, R/I is not $\mathscr{C}(P)$-torsion. (Here, when $\mathscr{C}$ is not *a priori* an Ore set, we say a module M is $\mathscr{C}$-torsion if for every $m\in M$ there is a $c\in\mathscr{C}$ with $mc=0$.) Let I be a right ideal and B its bound (i.e., $B=\text{r-ann}(R/I)$). If R/I is not $\mathscr{C}(X)$-torsion, then certainly R/B is not, since R/I is a homomorphic image of R/B. Hence, $B\cap\mathscr{C}(X)=\varnothing$, so, by hypothesis, $B\cap\mathscr{C}(P)=\varnothing$ for some P, $P\in X$, so R/B is not $\mathscr{C}(P)$-torsion.

Because R is FBN, R/B imbeds as a right submodule in a finite direct sum of copies of R/I, so R/I cannot be $\mathscr{C}(P)$-torsion. This establishes the result.

LEMMA 11. *Let R be a right FBN ring and $\mathscr{S}$ a right Ore set in R. Then there is a right localizable set X of prime ideals in R such that if $\mathscr{C} = \mathscr{C}(X)$, then $\mathscr{C} \supseteq \mathscr{S}$ and $R\mathscr{S}^{-1} = R\mathscr{C}^{-1}$.*

Proof. Let X be the set of primes in R which are contractions of maximal ideals in $R\mathscr{S}^{-1}$. It is clear that $\mathscr{C}(X) \supseteq \mathscr{S}$ and that the elements of $\mathscr{C}(X)$ all become units in $R\mathscr{S}^{-1}$. It is also clear that $R\mathscr{S}^{-1}$ is a right quotient ring for R with respect to the set $\mathscr{C}(X)$, from which it follows that $\mathscr{C}(X)$ is right Ore and $R\mathscr{S}^{-1} = R\mathscr{C}^{-1}$. (Alternatively, one can show that $\mathscr{C}(X)$ is right Ore by noting that for $c \in \mathscr{C}$, $c^{-1} = r_0 s_0^{-1}$ for some $s_0 \in \mathscr{S}$. Given an $r \in R$, find elements $r' \in R$ and $s' \in S$ with $s_0 r' = rs'$, and then note that $cr_0 r' = rs'$, thus verifying the right Ore condition for $\mathscr{C}$.) The way X was chosen makes it clear that condition (ii) in the definition of localizable sets of primes is satisfied.

Proof of Theorem B. Let $\mathscr{S}$ be a right Ore set in R. According to Lemma 11, there is a set X of incomparable prime ideals such that if $\mathscr{C} = C(X)$, then $\mathscr{C} \supseteq \mathscr{S}$, X is right localizable, and $R\mathscr{S}^{-1} = R\mathscr{C}^{-1}$. To show that $\mathscr{C}$ is a two-sided Ore set, we apply Lemma 10, where condition (ii) is already satisfied. Thus, we must show that if $P \in X$ and $P \rightsquigarrow Q$ then $Q \in X$. Since X is right link closed, it follows from Theorem D that X is also left link closed. This completes the proof.

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